

# Estimators for a Class of Bivariate Measures of Concordance for Copulas

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## Abstract

In the present paper we propose and study estimators for a wide class of bivariate measures of concordance for copulas. These measures of concordance are generated by a copula and generalize Spearman's rho and Gini's gamma. In the case of Spearman's rho and Gini's gamma the estimators turn out to be the usual sample versions of these measures of concordance.

## 1 Introduction

The history of measures of concordance (or measures of association) starts with measures of concordance for a sample of bivariate random vectors. Later, related measures of concordance were introduced for bivariate distribution functions and copulas, and the sample versions for random vectors were interpreted as estimators of the population versions for distribution functions or copulas. Moreover, axioms for bivariate measures of concordance for copulas were developed, and most of these concepts have been extended to the multivariate case, with particular emphasis on Kendall's tau, Spearman's rho and Gini's gamma.

In the present paper we propose and study estimators for a wide class of bivariate measures of concordance for copulas. These measures of concordance are generated by a copula and generalize Spearman's rho and Gini's gamma. In the case of Spearman's rho and Gini's gamma the estimators turn out to be the usual sample versions of these measures of concordance.

This paper is organized as follows: In Section 2 we resume some results on a group of transformations of copulas, invariance of copulas under a subgroup, measures of concordance for copulas which are defined in terms of the group, and a biconvex form for copulas. In Section 3 we consider a class of bivariate measures of concordance which are defined in terms of the biconvex form and are generated by a copula which is invariant under the full group of transformations; this class contains Spearman's rho and Gini's gamma as well as certain interpolations as special cases. In Section 4 we use the empirical copula to construct an estimator of the value of such a measure of concordance when the copula to be measured is unknown. To complete the discussion, we conclude with an Appendix on estimation under partial information on the copula to be measured: If the copula is known to be invariant under a specific subgroup of the group of transformations, then the value of every measure of concordance is equal to zero and the estimation problem is void.

We denote by  $\mathbf{0}$  the vector in  $\mathbb{R}^d$  whose coordinates are all equal to 0 and by  $\mathbf{1}$  the vector in  $\mathbb{R}^d$  whose coordinates are all equal to 1. For a set  $B \subseteq \mathbb{R}^d$ , the indicator function  $\chi_B : \mathbb{R}^d \rightarrow \{0, 1\}$  is defined by  $\chi_B(\mathbf{x}) := 1$  if  $\mathbf{x} \in B$  and  $\chi_B(\mathbf{x}) := 0$  else.

## 2 Preliminaries

In this section, we recall some definitions and results for the general dimension  $d \geq 2$  and point out the particularities in the case  $d = 2$  which are important for the subject of this paper.

### A group of transformations of copulas

Let  $\mathcal{C}$  denote the collection of all copulas  $[0, 1] \rightarrow [0, 1]$ . A map  $\varphi : \mathcal{C} \rightarrow \mathcal{C}$  is said to be a *transformation* on  $\mathcal{C}$ . Let  $\Phi$  denote the collection of all transformations on  $\mathcal{C}$  and define the *composition*  $\circ : \Phi \times \Phi \rightarrow \Phi$  by letting  $(\varphi \circ \psi)(C) := \varphi(\psi(C))$ . The composition is associative and the transformation  $\iota \in \Phi$  given by  $\iota(C) := C$  satisfies  $\iota \circ \varphi = \varphi = \varphi \circ \iota$  for every  $\varphi \in \Phi$  and is therefore called the *identity* on  $\mathcal{C}$ . Thus,  $(\Phi, \circ)$  is a semigroup with neutral element  $\iota$ .

We now introduce two types of elementary transformations. To this end, let  $\mathcal{M}$  denote the collection of all functions  $[0, 1] \rightarrow \mathbb{R}$ . For  $i, j \in \{1, \dots, d\}$  such that  $i \neq j$  we define the *transposition*  $\pi_{i,j} : \mathcal{C} \rightarrow \mathcal{M}$  by letting

$$(\pi_{i,j}(C))(\mathbf{u}) := \begin{cases} C(u_1, \dots, u_{i-1}, u_j, \dots, u_i, u_{j+1}, \dots, u_d) & \text{if } i < j \\ C(u_1, \dots, u_{j-1}, u_i, \dots, u_j, u_{i+1}, \dots, u_d) & \text{if } i > j \end{cases}$$

and for  $k \in \{1, \dots, d\}$  we define the *partial reflection*  $\nu_k : \mathcal{C} \rightarrow \mathcal{M}$  by letting

$$(\nu_k(C))(\mathbf{u}) := C(u_1, \dots, u_{k-1}, 1, u_{k+1}, \dots, u_d) - C(u_1, \dots, u_{k-1}, 1 - u_k, u_{k+1}, \dots, u_d)$$

Every transposition and every partial reflection is an involution in  $\Phi$ , and there exists a smallest subgroup  $\Gamma$  of  $\Phi$  containing all transpositions and all partial reflections. The group  $\Gamma$  is a representation of the hyperoctahedral group with  $d! 2^d$  elements.

A transformation is called

- a *permutation* if it can be expressed as a finite composition of transpositions, and it is called
- a *reflection* if it can be expressed as a finite composition of partial reflections.

Every transformation in  $\Gamma$  can be expressed as a composition of a permutation and a reflection. We denote by

- $\Gamma^\pi$  the set of all permutations and by
- $\Gamma^\nu$  the set of all reflections.

Then  $\Gamma^\pi$  and  $\Gamma^\nu$  are subgroups of  $\Gamma$ , and  $\Gamma^\nu$  is commutative while  $\Gamma^\pi$  is not.

Among the reflections, the *total reflection*

$$\tau := \nu_d \circ \dots \circ \nu_1$$

is of particular importance. The total reflection is an involution which transforms every copula into its survival copula. Define

- $\Gamma^\tau := \{\iota, \tau\}$  and
- $\Gamma^{\pi, \tau} := \{\gamma \in \Gamma \mid \gamma = \pi \circ \varphi \text{ for some } \pi \in \Gamma^\pi \text{ and some } \varphi \in \Gamma^\tau\}$

Then  $\Gamma^\tau$  is the center of  $\Gamma$ , and  $\Gamma^{\pi,\tau}$  is a subgroup of  $\Gamma$ . The total reflection also generates an order relation on  $\mathcal{C}$  which compares not only two copulas but also their survival copulas: For  $C, D \in \mathcal{C}$  we write  $C \preceq_\tau D$  if  $C \leq D$  and  $\tau(C) \leq \tau(D)$ . Then  $\preceq_\tau$  is an order relation on  $\mathcal{C}$  which is called the *concordance order*.

**2.1 Remark (Bivariate case).** Assume that  $d = 2$  and let

$$\pi := \pi_{1,2}$$

Then we have  $\nu_2 = \pi \circ \nu_1 \circ \pi$  and

$$\begin{aligned}\Gamma^\pi &= \{\iota, \pi\} \\ \Gamma^{\pi,\tau} &= \{\iota, \pi, \tau, \pi \circ \tau\} \\ \Gamma^\nu &= \{\iota, \nu_1, \nu_2, \tau\}\end{aligned}$$

Moreover,  $\Gamma$  is the smallest subgroup of  $\Phi$  containing  $\pi$  and  $\nu_1$ , and the concordance order  $\preceq_\tau$  coincides with the pointwise order  $\leq$  on  $\mathcal{C}$ .

Proofs and further details on the group of transformations may be found in Fuchs and Schmidt [2014] ( $d = 2$ ) and in Fuchs [2014]. We note in passing that in Fuchs and Schmidt [2014] the symbol  $\nu$  is used instead of  $\tau$ .

## Invariance of copulas with respect to a subgroup

For a subgroup  $\Lambda \subseteq \Gamma$ , a copula  $C$  is said to be  $\Lambda$ -invariant if it satisfies  $\gamma(C) = C$  for every  $\gamma \in \Lambda$ . For example, the product copula  $\Pi$  is  $\Gamma$ -invariant, and the upper Fréchet–Hoeffding bound  $M$  is  $\Gamma^{\pi,\tau}$ -invariant. Moreover, for every copula  $C$  and any subgroup  $\Lambda \subseteq \Gamma$ , the mean

$$C_\Lambda := \frac{1}{|\Lambda|} \sum_{\gamma \in \Lambda} \gamma(C)$$

is a copula since  $\mathcal{C}$  is convex, and  $C_\Lambda$  is  $\Lambda$ -invariant since  $\Lambda$  is a subgroup. The collection of all  $\Lambda$ -invariant copulas is convex.

**2.2 Remark (Bivariate case).** Assume that  $d = 2$ . Then the lower Fréchet–Hoeffding bound  $W = \nu_1(M)$  is a copula which is  $\Gamma^{\pi,\tau}$ -invariant.

Proofs and further details on invariance of copulas may be found in Fuchs and Schmidt [2014] ( $d = 2$ ) and in Fuchs [2016b].

## Measures of concordance for copulas

A map  $\kappa : \mathcal{C} \rightarrow \mathbb{R}$  is said to be a *measure of concordance* if it satisfies the following axioms:

- (i)  $\kappa[M] = 1$ .
- (ii) The identity  $\kappa[\gamma(C)] = \kappa[C]$  holds for every  $\gamma \in \Gamma^{\pi,\tau}$  and for all  $C \in \mathcal{C}$ .
- (iii) The identity  $\sum_{\nu \in \Gamma^\nu} \kappa[\nu(C)] = 0$  holds for all  $C \in \mathcal{C}$ .

These axioms are part of those proposed by Taylor [2007]. They imply that for every  $\Gamma^\nu$ -invariant copula  $C$ , and hence for every  $\Gamma$ -invariant copula and in particular the product copula, the identity  $\kappa[C] = 0$  holds for every measure of concordance  $\kappa$ .

A measure of concordance  $\kappa : \mathcal{C} \rightarrow \mathbb{R}$  is said to be

- *convex* if  $\kappa[(1-q)C + qD] = (1-q)\kappa[C] + q\kappa[D]$  holds for all  $C, D \in \mathcal{C}$  and all  $q \in (0, 1)$ .
- *order preserving* if  $\kappa[C] \leq \kappa[D]$  holds whenever  $C, D \in \mathcal{C}$  satisfy  $C \leq D$ .
- *concordance order preserving* if  $\kappa[C] \leq \kappa[D]$  holds whenever  $C, D \in \mathcal{C}$  satisfy  $C \preceq_\tau D$ .
- *continuous* if  $\lim_{n \rightarrow \infty} \kappa[C_n] = \kappa[C]$  holds for any sequence  $\{C_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}$  and any copula  $C \in \mathcal{C}$  such that  $\lim_{n \rightarrow \infty} C_n = C$  pointwise.

If  $\kappa$  is a concordance order preserving measure of concordance, then  $\kappa[C] \leq \kappa[M] = 1$  holds for all  $C \in \mathcal{C}$ .

**2.3 Remark (Bivariate case).** Assume that  $d = 2$ . Then a map  $\kappa : \mathcal{C} \rightarrow \mathbb{R}$  is a measure of concordance if and only if it has the following properties:

- (i)  $\kappa[M] = 1$ .
- (ii) The identity  $\kappa[\pi(C)] = \kappa[C]$  holds for all  $C \in \mathcal{C}$ .
- (iii) The identity  $\kappa[\nu_1(C)] = -\kappa[C]$  holds for all  $C \in \mathcal{C}$ .

Moreover, a measure of concordance  $\kappa$  is concordance order preserving if and only if it is order preserving, and in this case  $-1 = \kappa[W] \leq \kappa[C] \leq \kappa[M] = 1$  holds for all  $C \in \mathcal{C}$ .

Proofs and further details on measures of concordance (as defined above) may be found in Fuchs and Schmidt [2014] ( $d = 2$ ) and in Fuchs [2016b].

## A biconvex form for copulas

Consider the map  $[\cdot, \cdot] : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$  given by

$$[C, D] := \int_{[0,1]} C(\mathbf{u}) dQ^D(\mathbf{u})$$

where  $Q^D$  denotes the probability measure associated with the copula  $D$ ; see Fuchs [2016a]. The map  $[\cdot, \cdot]$  is in either argument linear with respect to convex combinations and is therefore called a *biconvex form*. Moreover, the map  $[\cdot, \cdot]$  is in either argument monotonically increasing with respect to the concordance order  $\preceq_\tau$  and it satisfies  $0 \leq [C, D] \leq [M, M] = 1/2$  for all  $C, D \in \mathcal{C}$ . Furthermore, there exist copulas  $C, D \in \mathcal{C}$  such that  $[C, D] = 0$ , and if  $C, D \in \mathcal{C}$  are  $\Gamma^\nu$ -invariant, then  $[C, D] = 1/2^d = [\Pi, \Pi]$ . Later on, we will also use the identities  $[M, \Pi] = 1/(d+1)$  and  $[M, M_{\Gamma^\nu}] = 1/4 + (1/2)^{d+1}$ .

**2.4 Remark (Bivariate case).** The biconvex form  $[\cdot, \cdot]$  is symmetric if and only if  $d = 2$ .

Proofs and further details on the biconvex form may be found in Fuchs [2016a].

## 3 Measures of concordance generated by a copula

In the remainder of this paper we confine ourselves to the bivariate case  $d = 2$  and we consider a wide class of measures of concordance which are defined in terms of the biconvex form.

Consider a fixed copula  $A \in \mathcal{C}$ . Then we have  $[M, A] > [\Pi, A]$  such that the map  $\kappa_A : \mathcal{C} \rightarrow \mathbb{R}$  given by

$$\kappa_A[C] := \frac{[C, A] - [\Pi, A]}{[M, A] - [\Pi, A]}$$

is well-defined; see Fuchs [2016b]. We have the following result (see Edwards et al. [2005], Behboodian et al. [2005], Fuchs and Schmidt [2014], Fuchs [2016b]):

### 3.1 Proposition.

- (1) *The map  $\kappa_A$  is a measure of concordance if and only if  $A$  is  $\Gamma$ -invariant.*
- (2) *If  $A$  is  $\Gamma$ -invariant, then  $\kappa_A$  is convex, order preserving and continuous.*
- (3) *If  $A$  is  $\Gamma$ -invariant, then  $[\Pi, A] = 1/4$  and the identity*

$$\kappa_A[C] = \frac{[C, A] - 1/4}{[M, A] - 1/4}$$

*holds for all  $C \in \mathcal{C}$ .*

### 3.2 Examples.

- (1) **Spearman's rho:** The copula  $\Pi$  is  $\Gamma$ -invariant and  $\kappa_\Pi$  satisfies

$$\kappa_\Pi[C] = 12[C, \Pi] - 3$$

which means that  $\kappa_\Pi$  is Spearman's rho; see Fuchs and Schmidt [2014] and Nelsen [2006; Subsection 5.1.2].

- (2) **Gini's gamma:** The copula  $M_{\Gamma^\nu} = (1/2)(M+W)$  is  $\Gamma$ -invariant and  $\kappa_{M_{\Gamma^\nu}}$  satisfies

$$\kappa_{M_{\Gamma^\nu}}[C] = 8[C, M_{\Gamma^\nu}] - 2$$

which means that  $\kappa_{M_{\Gamma^\nu}}$  is Gini's gamma; see Fuchs and Schmidt [2014] and Nelsen [2006; Subsection 5.1.4].

- (3) **Linear interpolation:** For  $q \in [0, 1]$  define

$$E_q := (1-q)\Pi + qM_{\Gamma^\nu}$$

Then  $E_q$  is a  $\Gamma$ -invariant copula and the measure of concordance  $\kappa_{E_q}$  satisfies

$$\begin{aligned} \kappa_{E_q}[C] &= \frac{[C, E_q] - 1/4}{[M, E_q] - 1/4} \\ &= \frac{(1-q)([C, \Pi] - 1/4) + q([C, M_{\Gamma^\nu}] - 1/4)}{(1-q)([M, \Pi] - 1/4) + q([M, M_{\Gamma^\nu}] - 1/4)} \\ &= \frac{(1-q)([M, \Pi] - 1/4)}{(1-q)([M, \Pi] - 1/4) + q([M, M_{\Gamma^\nu}] - 1/4)} \kappa_\Pi[C] \\ &\quad + \frac{q([M, M_{\Gamma^\nu}] - 1/4)}{(1-q)([M, \Pi] - 1/4) + q([M, M_{\Gamma^\nu}] - 1/4)} \kappa_{M_{\Gamma^\nu}}[C] \\ &= \frac{2(1-q)}{2+q} \kappa_\Pi[C] + \frac{3q}{2+q} \kappa_{M_{\Gamma^\nu}}[C] \end{aligned}$$

Therefore,  $\kappa_{E_q}$  is a weighted mean of Spearman's rho and Gini's gamma, and for  $q \in (0, 1)$  the respective weights are distinct from  $1-q$  and  $q$ .

The last example can be extended to the case of arbitrary  $\Gamma$ -invariant copulas in the place of  $\Pi$  and  $M_{\Gamma^\nu}$ .

## 4 Estimation

We still assume that  $d = 2$ , and we also assume henceforth that the copula  $A$  is  $\Gamma$ -invariant.

For an arbitrary and unknown copula  $C$ , our aim is to construct an estimator of  $\kappa_A[C]$  on the basis of a sample of bivariate random vectors with sample size  $n \geq 2$ . This estimator is based on an appropriate definition of the empirical copula, which in turn relies on the relative rank transform.

The relative rank transform is constructed in three steps:

- Consider first the *order transform*  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is defined coordinatewise by letting

$$(T(\mathbf{x}))_k := \min_{J \subseteq \{1, \dots, n\}, |J|=k} \max_{j \in J} x_j$$

for all  $k \in \{1, \dots, n\}$ ; see e.g. Fuchs and Schmidt [2016]. The map  $T$  is measurable and for every  $\mathbf{x} \in \mathbb{R}^n$  the coordinates of  $T(\mathbf{x})$  are increasing but need not be distinct.

- Consider next the *rank transform*  $R : \mathbb{R}^n \rightarrow \{1, \dots, n\}$  which is defined coordinatewise as follows: Let  $k_1 := \min\{k \in \{1, \dots, n\} | x_k = (T(\mathbf{x}))_1\}$  and define

$$(R(\mathbf{x}))_{k_1} := 1$$

and for  $i \in \{2, \dots, n\}$  let  $k_i := \min\{k \in \{1, \dots, n\} \setminus \{k_1, \dots, k_{i-1}\} | x_k = (T(\mathbf{x}))_i\}$  and define

$$(R(\mathbf{x}))_{k_i} := i$$

The map  $R$  is measurable and onto, and it satisfies  $\sum_{k=1}^n (R(\mathbf{x}))_k = n(n+1)/2$ .

- Consider finally the *relative rank transform*  $U : \mathbb{R}^n \rightarrow \{1/(n+1), \dots, n/(n+1)\}$  given by

$$U(\mathbf{x}) := \frac{1}{n+1} R(\mathbf{x})$$

The map  $U$  is measurable and onto, and it satisfies  $\sum_{k=1}^n (U(\mathbf{x}))_k = n/2$ .

Consider now a probability space  $(\Omega, \mathcal{F}, P)$  and an i.i.d. family  $\{\mathbf{X}_k\}_{k \in \{1, \dots, n\}}$  of random vectors  $\Omega \rightarrow \mathbb{R}^2$  such that  $C$  is a copula for the distribution function of every  $\mathbf{X}_k$ . The family  $\{\mathbf{X}_k\}_{k \in \{1, \dots, n\}}$  can be represented by the random matrix

$$\mathbf{X}_{(n)} := (\mathbf{X}_1, \dots, \mathbf{X}_n) = \begin{pmatrix} X_{1,1} & \dots & X_{1,n} \\ X_{2,1} & \dots & X_{2,n} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'_{(n,1)} \\ \mathbf{X}'_{(n,2)} \end{pmatrix}$$

with  $(X_{1,k}, X_{2,k}) := \mathbf{X}'_k$  and  $\mathbf{X}_{(n,i)} := (X_{i,1}, \dots, X_{i,n})'$  for all  $k \in \{1, \dots, n\}$  and  $i \in \{1, 2\}$ . Define now

$$\mathbf{U}_{(n)} := \begin{pmatrix} (U \circ \mathbf{X}_{(n,1)})' \\ (U \circ \mathbf{X}_{(n,2)})' \end{pmatrix} = \begin{pmatrix} U_{1,1} & \dots & U_{1,n} \\ U_{2,1} & \dots & U_{2,n} \end{pmatrix} = (\mathbf{U}_1, \dots, \mathbf{U}_n)$$

with  $(U_{i,1}, \dots, U_{i,n}) := (U \circ \mathbf{X}_{(n,i)})'$  and  $\mathbf{U}_k := (U_{1,k}, U_{2,k})'$  for all  $i \in \{1, 2\}$  and  $k \in \{1, \dots, n\}$ . Then the rows of the random matrix  $\mathbf{U}_{(n)}$  contain the relative ranks (without repetition) of the rows of the random matrix  $\mathbf{X}_{(n)}$ .

The map  $\widehat{C}_{(n)} : [\mathbf{0}, \mathbf{1}] \times \Omega \rightarrow [0, 1]$  given by

$$\widehat{C}_{(n)}(\mathbf{u}, \omega) := \frac{1}{n} \sum_{k=1}^n \chi_{[\mathbf{0}, \mathbf{u}]}(\mathbf{U}_k(\omega)) = \frac{1}{n} \sum_{k=1}^n \chi_{[\mathbf{U}_k(\omega), \mathbf{1}]}(\mathbf{u})$$

is called the *empirical copula* with sample size  $n$  (although it is not a copula since it fails to be continuous). This definition is appropriate for our purpose but it differs from that used in Nelsen [2006; Section 5.6].

**4.1 Lemma.** *The empirical copula satisfies*

$$\int_{[\mathbf{0}, \mathbf{1}]} \widehat{C}_{(n)}(\mathbf{u}, \omega) dQ^A(\mathbf{u}) = \frac{1}{n} \sum_{k=1}^n A(\mathbf{U}_k(\omega))$$

for every  $\omega \in \Omega$ .

*Proof.* By continuity of  $A$  and because of the identity  $\sum_{k=1}^n U_{i,k}(\omega) = n/2$  for  $i \in \{1, 2\}$ , we obtain

$$\begin{aligned} \int_{[\mathbf{0}, \mathbf{1}]} \widehat{C}_{(n)}(\mathbf{u}, \omega) dQ^A(\mathbf{u}) &= \int_{[\mathbf{0}, \mathbf{1}]} \frac{1}{n} \sum_{k=1}^n \chi_{[\mathbf{U}_k(\omega), \mathbf{1}]}(\mathbf{u}) dQ^A(\mathbf{u}) \\ &= \frac{1}{n} \sum_{k=1}^n \int_{[\mathbf{0}, \mathbf{1}]} \chi_{(\mathbf{U}_k(\omega), \mathbf{1}]}(\mathbf{u}) dQ^A(\mathbf{u}) \\ &= \frac{1}{n} \sum_{k=1}^n Q^A[(\mathbf{U}_k(\omega), \mathbf{1}]] \\ &= \frac{1}{n} \sum_{k=1}^n \left( 1 - A(U_{1,k}(\omega), 1) - A(1, U_{2,k}(\omega)) + A(U_{1,k}(\omega), U_{2,k}(\omega)) \right) \\ &= \frac{1}{n} \sum_{k=1}^n \left( 1 - U_{1,k}(\omega) - U_{2,k}(\omega) + A(U_{1,k}(\omega), U_{2,k}(\omega)) \right) \\ &= \frac{1}{n} \sum_{k=1}^n A(U_{1,k}(\omega), U_{2,k}(\omega)) \end{aligned}$$

as was to be shown. □

Because of the previous result, we use the random variable

$$\langle C, A \rangle_{(n)} := \frac{1}{n} \sum_{k=1}^n A \circ \mathbf{U}_k$$

as an estimator of  $[C, A]$  when nothing is known about the copula  $C$ . By contrast, if it is known that  $C = M$ , then the coordinates of every  $\mathbf{X}_k$  are comonotone and the relative ordinal ranks satisfy  $U_{1,k} = U_{2,k} = k/(n+1)$  almost surely. Therefore, we use the real number

$$\langle M, A \rangle_{(n)} := \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{k}{n+1}\right)$$

as an estimator of  $[M, A]$ . Correspondingly, we use the real number

$$\langle W, A \rangle_{(n)} := \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{n+1-k}{n+1}\right)$$

as an estimator of  $[W, A]$ .

#### 4.2 Lemma.

- (1)  $\langle W, A \rangle_{(n)} \leq \langle C, A \rangle_{(n)} \leq \langle M, A \rangle_{(n)}$ .
- (2)  $\langle M, A \rangle_{(n)} + \langle W, A \rangle_{(n)} = 1/2$ .
- (3) If  $\langle M, A \rangle_{(n)} = 1/4$ , then  $Q^A[(1/(n+1), n/(n+1))^2] = 0$ .
- (4) If  $Q^A[(1/(n+1), n/(n+1))^2] > 0$ , then  $\langle M, A \rangle_{(n)} > 1/4$ .

*Proof.* To prove (1), consider a realization

$$(\mathbf{u}_1, \dots, \mathbf{u}_n) = \begin{pmatrix} u_{1,1} & \dots & u_{1,n} \\ u_{2,1} & \dots & u_{2,n} \end{pmatrix}$$

of the random matrix  $(\mathbf{U}_1, \dots, \mathbf{U}_n)$ . Then every row of the matrix

$$\begin{pmatrix} u_{1,1} & \dots & u_{1,n} \\ u_{2,1} & \dots & u_{2,n} \end{pmatrix}$$

contains each of the real numbers  $1/(n+1), \dots, n/(n+1)$  exactly once. Put

$$\mathbf{u}_k^{(0)} := \mathbf{u}_k$$

and for  $p \in \{1, \dots, n\}$  proceed as follows: Consider the unique  $l_p \in \{1, \dots, n+1-p\}$  for which

$$\mathbf{u}_{l_p}^{(p-1)} = \left( \frac{i}{n+1}, \frac{n+1-p}{n+1} \right)'$$

holds for some  $i \in \{1, \dots, n\}$ .

– If  $i = n+1-p$ , put

$$\mathbf{u}_k^{(p)} := \mathbf{u}_k^{(p-1)}$$

for every  $k \in \{1, \dots, n\}$ .

– If  $i \leq n-p$ , consider the unique  $m_p \in \{1, \dots, n\}$  for which

$$\mathbf{u}_{m_p} = \left( \frac{n+1-p}{n+1}, \frac{j}{n+1} \right)'$$

holds for some  $j \in \{1, \dots, n\}$  and put

$$\mathbf{u}_k^{(p)} := \begin{cases} \left( \frac{n+1-p}{n+1}, \frac{n+1-p}{n+1} \right)' & \text{if } k = l_p \\ \left( \frac{i}{n+1}, \frac{j}{n+1} \right)' & \text{if } k = m_p \\ \mathbf{u}_k^{(p-1)} & \text{else} \end{cases}$$



In either case, we obtain

$$\mathbf{u}_{l_r}^{(p)} = \left( \frac{n+1-r}{n+1}, \frac{n+1-r}{n+1} \right)'$$

for all  $r \in \{1, \dots, p\}$ , and since  $A$  is 2-increasing we also obtain

$$\frac{1}{n} \sum_{k=1}^n A(\mathbf{u}_k^{(p-1)}) \leq \frac{1}{n} \sum_{k=1}^n A(\mathbf{u}_k^{(p)})$$

After  $n$  steps we thus obtain

$$\frac{1}{n} \sum_{k=1}^n A(\mathbf{u}_k) = \frac{1}{n} \sum_{k=1}^n A(\mathbf{u}_k^{(0)}) \leq \frac{1}{n} \sum_{k=1}^n A(\mathbf{u}_k^{(n)}) = \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{k}{n+1}\right)$$

and hence  $\langle C, A \rangle_{(n)} \leq \langle M, A \rangle_{(n)}$ . A similar algorithm yields  $\langle W, A \rangle_{(n)} \leq \langle C, A \rangle_{(n)}$ . This proves (1).

Since  $A$  is  $\Gamma$ -invariant, we have

$$A(u, u) = (\nu_2(A))(u, u) = A(u, 1) - A(u, 1-u) = u - A(u, 1-u)$$

and hence

$$\begin{aligned} \langle M, A \rangle_{(n)} &= \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{k}{n+1}\right) \\ &= \frac{1}{n} \sum_{k=1}^n \left( \frac{k}{n+1} - A\left(\frac{k}{n+1}, 1 - \frac{k}{n+1}\right) \right) \\ &= \frac{1}{2} - \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{n+1-k}{n+1}\right) \\ &= \frac{1}{2} - \langle W, A \rangle_{(n)} \end{aligned}$$

This proves (2).

Assume now that  $\langle M, A \rangle_{(n)} = 1/4$ . Because of (2), this yields

$$\begin{aligned} 0 &= \langle M, A \rangle_{(n)} - \langle W, A \rangle_{(n)} \\ &= \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{k}{n+1}\right) - \frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n+1}, \frac{n+1-k}{n+1}\right) \\ &= \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \left( A\left(\frac{n+1-k}{n+1}, \frac{n+1-k}{n+1}\right) - A\left(\frac{n+1-k}{n+1}, \frac{k}{n+1}\right) \right. \\ &\quad \left. - A\left(\frac{k}{n+1}, \frac{n+1-k}{n+1}\right) + A\left(\frac{k}{n+1}, \frac{k}{n+1}\right) \right) \end{aligned}$$

Since  $A$  is 2-increasing, every term under the last sum is nonnegative and hence equal to 0.

This proves (3).

Assume finally that  $Q^A[(1/(n+1), n/(n+1)]^2] > 0$ . Because of (3), this yields  $\langle M, A \rangle_{(n)} \neq 1/4$ , and it then follows from (1) and (2) that  $\langle M, A \rangle_{(n)} > 1/4$ .  $\square$

Since the sequence  $\{(1/(n+1), n/(n+1))^2\}_{n \in \mathbb{N}}$  is increasing with union  $(0, 1)^2$  and since  $Q^A[(0, 1)^2] = Q^A[[0, 1]] = 1$ , we see that there exists some  $n_A \in \mathbb{N}$  such that  $Q^A[(1/(n+1), n/(n+1))^2] > 0$ , and hence  $\langle M, A \rangle_{(n)} > 1/4$ , holds for every  $n \in \mathbb{N}$  with  $n \geq n_A$ . Since  $Q^A[(1/2, 1/2)^2] = Q^A[\emptyset] = 0$ , we have  $n_A \geq 2$ . The following example shows that  $n_A$  may be greater than 2:

**4.3 Example.** According to Nelson [2006; Formula (3.1.5)], the map  $E : [0, 1] \rightarrow [0, 1]$  given by

$$E(\mathbf{u}) := \begin{cases} M(u_1, u_2) & \text{if } |u_1 - u_2| > 1/2 \\ W(u_1, u_2) & \text{if } |u_1 + u_2 - 1| > 1/2 \\ \frac{u_1 + u_2}{2} - \frac{1}{4} & \text{else} \end{cases}$$

is a copula, and it is straightforward to prove that  $E$  is  $\Gamma$ -invariant and satisfies  $\langle M, E \rangle_{(3)} = 1/4$ . Now Lemma 4.2 yields  $Q^E[(1/4, 3/4)^2] = 0$ , and hence  $n_E \geq 4$ .

For the remainder of this section, we assume that the sample size  $n$  satisfies  $n \geq n_A$ . Then we have  $\langle M, A \rangle_{(n)} > 1/4$  and the random variable

$$\widehat{\kappa_A[C]}_{(n)} := \frac{\langle C, A \rangle_{(n)} - 1/4}{\langle M, A \rangle_{(n)} - 1/4}$$

is well-defined. We propose to use  $\widehat{\kappa_A[C]}_{(n)}$  as an estimator of  $\kappa_A[C]$ .

**4.4 Theorem.** *The estimator  $\widehat{\kappa_A[C]}_{(n)}$  satisfies*

$$-1 = \widehat{\kappa_A[W]}_{(n)} \leq \widehat{\kappa_A[C]}_{(n)} \leq \widehat{\kappa_A[M]}_{(n)} = 1$$

#### 4.5 Examples.

(1) **Spearman's rho:** Since

$$\langle C, \Pi \rangle_{(n)} = \frac{1}{n} \sum_{k=1}^n U_{1,k} U_{2,k}$$

and

$$\langle M, \Pi \rangle_{(n)} = \frac{1}{n} \sum_{k=1}^n \left( \frac{k}{n+1} \right)^2 = \frac{2n+1}{6(n+1)}$$

the estimator of Spearman's rho satisfies

$$\widehat{\kappa_\Pi[C]}_{(n)} = \frac{3}{n} \frac{n+1}{n-1} \left( 4 \sum_{k=1}^n U_{1,k} U_{2,k} - n \right)$$

Using the absolute ranks  $R_{i,k} := (n+1) U_{i,k}$  instead of the relative ranks  $U_{i,k}$ , the previous identity can be written as

$$\widehat{\kappa_\Pi[C]}_{(n)} = 1 - \frac{6}{n(n^2-1)} \sum_{k=1}^n (R_{1,k} - R_{2,k})^2$$

which shows that the estimator is just the sample version of Spearman's rho; see also Kruskal [1958], Joe [1990] and Pérez and Prieto-Alaiz [2016].

(2) **Gini's gamma:** Since

$$\begin{aligned}
\langle C, M_{\Gamma^\nu} \rangle_{(n)} &= \left\langle C, \frac{1}{2} (M+W) \right\rangle_{(n)} \\
&= \frac{1}{2} \left( \langle C, M \rangle_{(n)} + \langle C, W \rangle_{(n)} \right) \\
&= \frac{1}{2} \left( \frac{1}{n} \sum_{k=1}^n \min\{U_{1,k}, U_{2,k}\} + \frac{1}{n} \sum_{k=1}^n \max\{U_{1,k} + U_{2,k} - 1, 0\} \right)
\end{aligned}$$

and

$$\begin{aligned}
\langle M, M_{\Gamma^\nu} \rangle_{(n)} &= \left\langle M, \frac{1}{2} (M+W) \right\rangle_{(n)} \\
&= \frac{1}{2} \left( \langle M, M \rangle_{(n)} + \langle M, W \rangle_{(n)} \right) \\
&= \frac{1}{2} \left( \frac{1}{2} + \frac{1}{n} \sum_{k=\lfloor (n+2)/2 \rfloor}^n \left( \frac{2k}{n+1} - 1 \right) \right) \\
&= \frac{1}{4} + \frac{1}{4n(n+1)} \left\lfloor \frac{n^2}{2} \right\rfloor
\end{aligned}$$

the estimator of Gini's gamma satisfies

$$\widehat{\kappa_{M_{\Gamma^\nu}}[C]}_{(n)} = \frac{n+1}{\lfloor n^2/2 \rfloor} \left( 2 \sum_{k=1}^n \min\{U_{1,k}, U_{2,k}\} + 2 \sum_{k=1}^n \max\{U_{1,k} + U_{2,k} - 1, 0\} - n \right)$$

Straightforward although slightly tedious calculation yields

$$\widehat{\kappa_{M_{\Gamma^\nu}}[C]}_{(n)} = \frac{1}{\lfloor n^2/2 \rfloor} \left( \sum_{k=1}^n |R_{1,k} + R_{2,k} - 1| - \sum_{k=1}^n |R_{1,k} - R_{2,k}| \right)$$

which shows that the estimator is just the sample version of Gini's gamma; see Nelsen [2006; Subsection 5.1.4].

(3) **Linear interpolation:** For  $q \in [0, 1]$  consider the  $\Gamma$ -invariant copula

$$E_q = (1-q) \Pi + q M_{\Gamma^\nu}$$

introduced in Example 3.2(3). The estimator of  $\kappa_{E_q}[C]$  satisfies

$$\begin{aligned}
\widehat{\kappa_{E_q}}[C]_{(n)} &= \frac{\langle C, E_q \rangle_{(n)} - 1/4}{\langle M, E_q \rangle_{(n)} - 1/4} \\
&= \frac{(1-q)(\langle C, \Pi \rangle_{(n)} - 1/4) + q(\langle C, M_{\Gamma^\nu} \rangle_{(n)} - 1/4)}{(1-q)(\langle M, \Pi \rangle_{(n)} - 1/4) + q(\langle M, M_{\Gamma^\nu} \rangle_{(n)} - 1/4)} \\
&= \frac{(1-q)(\langle M, \Pi \rangle_{(n)} - 1/4)}{(1-q)(\langle M, \Pi \rangle_{(n)} - 1/4) + q(\langle M, M_{\Gamma^\nu} \rangle_{(n)} - 1/4)} \widehat{\kappa_{\Pi}}[C]_{(n)} \\
&\quad + \frac{q(\langle M, M_{\Gamma^\nu} \rangle_{(n)} - 1/4)}{(1-q)(\langle M, \Pi \rangle_{(n)} - 1/4) + q(\langle M, M_{\Gamma^\nu} \rangle_{(n)} - 1/4)} \widehat{\kappa_{M_{\Gamma^\nu}}}[C]_{(n)}
\end{aligned}$$

and hence is a weighted mean of the estimators of Spearman's rho and Gini's gamma; for  $q \in (0, 1)$  the respective weights are distinct from  $1 - q$  and  $q$ , due to the fact that  $\langle M, \Pi \rangle_{(n)} \neq \langle M, M_{\Gamma^\nu} \rangle_{(n)}$ .

The last example can be extended to the case of arbitrary  $\Gamma$ -invariant copulas in the place of  $\Pi$  and  $M_{\Gamma^\nu}$ .

## 5 Appendix

In certain cases in which some information on the copula  $C$  is available, there is no need to estimate  $\kappa[C]$  since this value is known. For example, the identities  $\kappa[W] = -1$ ,  $\kappa[\Pi] = 0$  and  $\kappa[M] = 1$  hold for every measure of concordance  $\kappa$ . Moreover, if the copula  $C$  is  $\Gamma^\nu$ -invariant, then the identity  $\kappa[C] = 0$  holds for every measure of concordance  $\kappa$  and the estimation problem for  $\kappa[C]$  is void. The following result provides a class of continuous distribution functions for which the unique copula is  $\Gamma^\nu$ -invariant:

**5.1 Theorem.** *Assume that  $C$  is a copula for which there exists a distribution function  $F : \mathbb{R}^2 \rightarrow [0, 1]$  with marginal distribution functions  $F_1, F_2 : \mathbb{R} \rightarrow [0, 1]$  and a measurable function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that*

$$f(x_1, x_2) = f(|x_1|, |x_2|)$$

and

$$C(F_1(x_1), F_2(x_2)) = F(x_1, x_2) = \int_{(-\infty, x_1] \times (-\infty, x_2]} f(\mathbf{s}) d\boldsymbol{\lambda}^2(\mathbf{s})$$

(with bivariate Lebesgue measure  $\boldsymbol{\lambda}^2$ ) holds for every  $\mathbf{x} \in \mathbb{R}^2$ . Then the copula  $C$  is  $\Gamma^\nu$ -invariant and the identity  $\kappa[C] = 0$  holds for every measure of concordance  $\kappa$ .

*Proof.* Consider  $\mathbf{u} \in [0, 1]$  and any  $\mathbf{x} \in \mathbb{R}^2$  satisfying  $F_i(x_i) = u_i$  for all  $i \in \{1, 2\}$ . Then we have

$$\begin{aligned} (\nu_1(C))(\mathbf{u}) &= C(1, u_2) - C(1 - u_1, u_2) \\ &= C(1, F_2(x_2)) - C(1 - F_1(x_1), F_2(x_2)) \\ &= C(1, F_2(x_2)) - C(F_1(-x_1), F_2(x_2)) \\ &= F_2(x_2) - F(-x_1, x_2) \\ &= \int_{(-\infty, \infty) \times (-\infty, x_2]} f(\mathbf{s}) d\boldsymbol{\lambda}^2(\mathbf{s}) - \int_{(-\infty, -x_1] \times (-\infty, x_2]} f(\mathbf{s}) d\boldsymbol{\lambda}^2(\mathbf{s}) \\ &= \int_{[-x_1, \infty) \times (-\infty, x_2]} f(\mathbf{s}) d\boldsymbol{\lambda}^2(\mathbf{s}) \\ &= \int_{(-\infty, x_1] \times (-\infty, x_2]} f(\mathbf{s}) d\boldsymbol{\lambda}^2(\mathbf{s}) \\ &= F(x_1, x_2) \\ &= C(F_1(x_1), F_2(x_2)) \\ &= C(\mathbf{u}) \end{aligned}$$

This yields  $\nu_1(C) = C$ , and repeating the argument yields  $\nu_2(C) = C$ . Therefore, the copula  $C$  is  $\Gamma^\nu$ -invariant.  $\square$

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